initial conditions are $y_0 = 0,$ and $v_0 = 200.0\text{ m/s}$.

We have, from **Equation 3.17**,

$$v^2 = (200.0\text{ m/s})^2 - 2(9.8\text{ m/s}^2)(1.0 \times 10^3\text{ m}) \Rightarrow v = \pm 142.8\text{ m/s}.$$

Significance

We have both a positive and negative solution in (b). Since our coordinate system has the positive direction upward, the +142.8 m/s corresponds to a positive upward velocity at 6000 m during the upward leg of the trajectory of the booster. The value $v = -142.8$ m/s corresponds to the velocity at 6000 m on the downward leg. This example is also important in that an object is given an initial velocity at the origin of our coordinate system, but the origin is at an altitude above the surface of Earth, which must be taken into account when forming the solution.

Visit **this site (https://openstaxcollege.org/l/21equatgraph)** to learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (for example, $y = bx$) to see how they add to generate the polynomial curve.

3.6 | Finding Velocity and Displacement from Acceleration

Learning Objectives

By the end of this section, you will be able to:

- Derive the kinematic equations for constant acceleration using integral calculus.
- Use the integral formulation of the kinematic equations in analyzing motion.
- Find the functional form of velocity versus time given the acceleration function.
- Find the functional form of position versus time given the velocity function.

This section assumes you have enough background in calculus to be familiar with integration. In **Instantaneous Velocity and Speed** and **Average and Instantaneous Acceleration** we introduced the kinematic functions of velocity and acceleration using the derivative. By taking the derivative of the position function we found the velocity function, and likewise by taking the derivative of the velocity function we found the acceleration function. Using integral calculus, we can work backward and calculate the velocity function from the acceleration function, and the position function from the velocity function.

Kinematic Equations from Integral Calculus

Let's begin with a particle with an acceleration a(t) which is a known function of time. Since the time derivative of the velocity function is acceleration,

$$\frac{d}{dt}v(t) = a(t),$$

we can take the indefinite integral of both sides, finding

$$\int \frac{d}{dt}v(t)dt = \int a(t)dt + C_1,$$

where C_1 is a constant of integration. Since $\int \frac{d}{dt}v(t)dt = v(t)$, the velocity is given by

$$v(t) = \int a(t)dt + C_1. \tag{3.18}$$

Similarly, the time derivative of the position function is the velocity function,

$$\frac{d}{dt}x(t) = v(t).$$

Thus, we can use the same mathematical manipulations we just used and find

$$x(t) = \int v(t)dt + C_2, \tag{3.19}$$

where C_2 is a second constant of integration.

We can derive the kinematic equations for a constant acceleration using these integrals. With $a(t) = a$ a constant, and doing the integration in **Equation 3.18**, we find

$$v(t) = \int a dt + C_1 = at + C_1.$$

If the initial velocity is $v(0) = v_0$, then

$$v_0 = 0 + C_1.$$

Then, $C_1 = v_0$ and

$$v(t) = v_0 + at,$$

which is **Equation 3.12**. Substituting this expression into **Equation 3.19** gives

$$x(t) = \int (v_0 + at)dt + C_2.$$

Doing the integration, we find

$$x(t) = v_0 t + \frac{1}{2}at^2 + C_2.$$

If $x(0) = x_0$, we have

$$x_0 = 0 + 0 + C_2;$$

so, $C_2 = x_0$. Substituting back into the equation for $x(t)$, we finally have

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2,$$

which is **Equation 3.13**.

Example 3.17

Motion of a Motorboat

A motorboat is traveling at a constant velocity of 5.0 m/s when it starts to decelerate to arrive at the dock. Its acceleration is $a(t) = -\frac{1}{4}t \text{ m/s}^3$. (a) What is the velocity function of the motorboat? (b) At what time does the velocity reach zero? (c) What is the position function of the motorboat? (d) What is the displacement of the motorboat from the time it begins to decelerate to when the velocity is zero? (e) Graph the velocity and position functions.

Strategy

(a) To get the velocity function we must integrate and use initial conditions to find the constant of integration. (b) We set the velocity function equal to zero and solve for t. (c) Similarly, we must integrate to find the position function and use initial conditions to find the constant of integration. (d) Since the initial position is taken to be zero, we only have to evaluate the position function at $t = 0$.

Solution

We take $t = 0$ to be the time when the boat starts to decelerate.

a. From the functional form of the acceleration we can solve **Equation 3.18** to get $v(t)$:

$$v(t) = \int a(t)dt + C_1 = \int -\frac{1}{4}t dt + C_1 = -\frac{1}{8}t^2 + C_1.$$

At $t = 0$ we have $v(0) = 5.0$ m/s $= 0 + C_1$, so $C_1 = 5.0$ m/s or $v(t) = 5.0\text{ m/s} - \frac{1}{8}t^2$.

b. $v(t) = 0 = 5.0\text{ m/s} - \frac{1}{8}t^2\text{ m/s}^3 \Rightarrow t = 6.3\text{ s}$

c. Solve **Equation 3.19**:

$$x(t) = \int v(t)dt + C_2 = \int (5.0 - \frac{1}{8}t^2)dt + C_2 = 5.0t\text{m/s} - \frac{1}{24}t^3\text{ m/s}^3 + C_2.$$

At $t = 0$, we set $x(0) = 0 = x_0$, since we are only interested in the displacement from when the boat starts to decelerate. We have

$$x(0) = 0 = C_2.$$

Therefore, the equation for the position is

$$x(t) = 5.0t - \frac{1}{24}t^3.$$

d. Since the initial position is taken to be zero, we only have to evaluate the position function at the time when the velocity is zero. This occurs at $t = 6.3$ s. Therefore, the displacement is

$$x(6.3) = 5.0(6.3\text{ s}) - \frac{1}{24}(6.3\text{ s}) = 21.1\text{ m}.$$

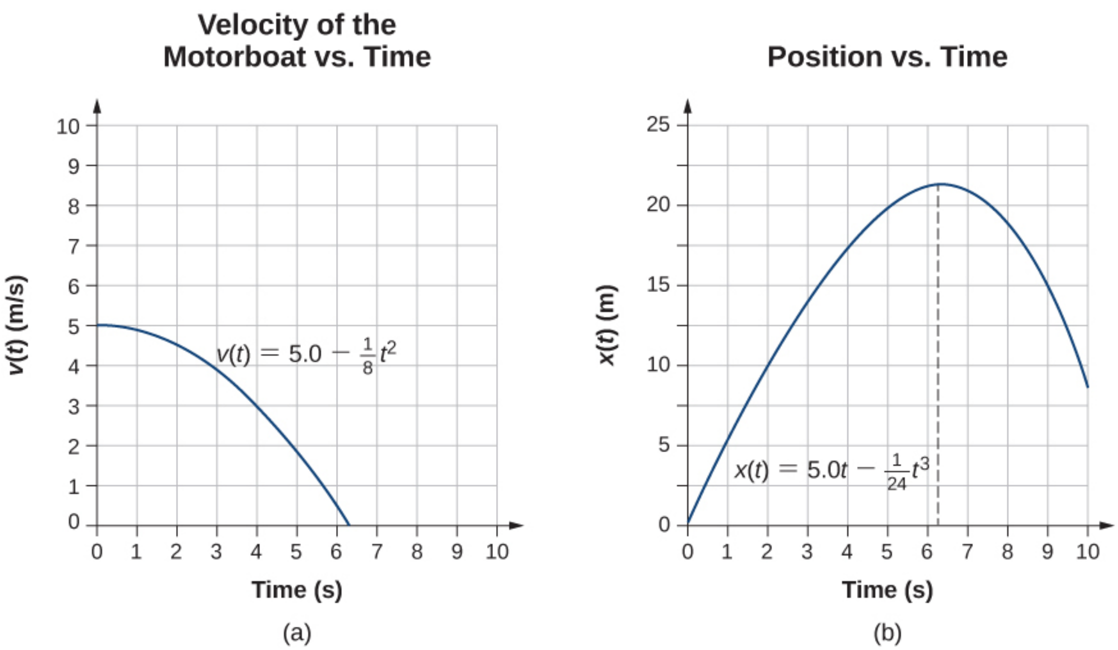

Figure 3.30 (a) Velocity of the motorboat as a function of time. The motorboat decreases its velocity to zero in 6.3 s. At times greater than this, velocity becomes negative—meaning, the boat is reversing direction. (b) Position of the motorboat as a function of time. At $t = 6.3$ s, the velocity is zero and the boat has stopped. At times greater than this, the velocity becomes negative—meaning, if the boat continues to move with the same acceleration, it reverses direction and heads back toward where it originated.

Significance

The acceleration function is linear in time so the integration involves simple polynomials. In **Figure 3.30**, we see that if we extend the solution beyond the point when the velocity is zero, the velocity becomes negative and the boat reverses direction. This tells us that solutions can give us information outside our immediate interest and we should be careful when interpreting them.

3.8 Check Your Understanding A particle starts from rest and has an acceleration function $5 - 10t \ \text{m/s}^2$. (a) What is the velocity function? (b) What is the position function? (c) When is the velocity zero?